Asymptotic axial symmetry of solutions of parabolic equations in bounded radial domains

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Abstract

We consider solutions of some nonlinear parabolic boundary value problems in radial bounded domains whose initial profile satisfy a reflection inequality with respect to a hyperplane containing the origin. We show that, under rather general assumptions, these solutions are asymptotically (in time) foliated Schwarz symmetric, i.e., all elements in the associated omega limit set are axially symmetric with respect to a common axis passing through the origin and nonincreasing in the polar angle from this axis. In this form, the result is new even for equilibria (i.e. solutions of the corresponding elliptic problem) and time periodic solutions.

1 Introduction

Consider the reaction-diffusion problem

$$
\begin{aligned}
u_t &= \Delta u + f(t, |x|, u), & (x,t) &\in (0, \infty) \times B, \\
u(x,t) &= 0, & (x,t) &\in \partial B \times (0, \infty), \\
u(x,0) &= u_0(x), & x &\in B,
\end{aligned}
\tag{1.1}
$$

where B is a bounded radial domain in $\mathbb{R}^N$, $N \geq 2$, i.e., a ball or an annulus in $\mathbb{R}^n$ centered at zero. If the nonlinearity f is continuous and locally Lipschitz in u uniformly in t and $|x|$, it follows from standard semigroup theory that, for every $u_0 \in C(\overline{B})$, the corresponding local (in time) problem admits a unique solution $u \in C(\overline{B} \times [0, T(u_0)))$ for some time $T(u_0) > 0$. Moreover, it has been studied extensively in recent years under which assumptions on the nonlinearity f and the initial condition u_0 this unique solution exists globally in time and

the corresponding orbit $\{u(\cdot, t) \ : \ t > 0\}$ is relatively compact in $C(\overline{B})$. (1.2)

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We refer the reader to [4, 13, 15] and the references therein, where many specific examples are discussed which give rise to this behavior. It is then natural to investigate the qualitative asymptotic behaviour of these global solutions. In this paper, we are mainly inspired by work of Poláčik [13] who studied asymptotic symmetry of these solutions. In particular, he proved that, if B is a ball, f is nonincreasing in $|x|$ and if f satisfies some rather mild regularity assumptions, every nonnegative solution of (1.1) satisfying (1.2) is asymptotically radially symmetric. More precisely, every function z belonging to the omega limit set

$$\omega(u) := \{z \in C(\overline{B}) : \|u(\cdot, t_n) - z\|_{L^\infty(B)} \to 0 \text{ for some } t_n \to \infty\}$$

is radially symmetric and decreasing in the radial variable (see [13, Corollary 2.6]). This result is proved via a parabolic version of the moving plane method relying on subtle estimates on solutions to linear parabolic equations. We recall that the moving plane method has its roots in earlier work of Alexandrov [1] and Serrin [16] for geometric problems and has been elaborated in the seminal paper of Gidas, Ni and Nirenberg [7] in order to prove symmetry results for solutions of elliptic nonlinear boundary problems.

The motivation of the present paper is that, to our knowledge, so far no asymptotic symmetry result is available for sign-changing solutions of (1.1) and, if $B \subset \mathbb{R}^n$ is an annulus or f is increasing in $|x|$, also for nonnegative solutions. In fact, in these situations, equilibrium solutions of (1.1) in the case where $f = f(|x|, u)$ does not depend on t may already have a very complicated shape. In particular, for suitable data, solutions with arbitrarily many isolated local maxima close to the boundary have been constructed, see [5, 10, 11]. Therefore any type of symmetry result in this setting requires additional assumptions on the initial profile u_0. In this paper we assume a simple reflection inequality with respect to a hyperplane. In order to state this assumption and our symmetry result, we need to introduce some notation. Let $S = \{x \in \mathbb{R}^N : |x| = 1\}$ be the unit sphere in $\mathbb{R}^N$. For a vector $e \in S$, we consider the hyperplane $H(e) := \{x \in \mathbb{R}^N : x \cdot e = 0\}$ and the half domain $B(e) := \{x \in B : x \cdot e > 0\}$. We write also $\sigma_e : B \to B$ to denote reflection with respect to $H(e)$, i.e. $\sigma_e(x) := x - 2(x \cdot e)e$ for each $x \in B$. We say that a function $u \in C(B)$ is *foliated Schwarz symmetric with respect to some unit vector* $p \in S$ if u is axially symmetric with respect to the axis $\mathbb{R}p$ and nonincreasing in the polar angle $\theta := \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$. The name *foliated Schwarz symmetry* was introduced in [17] by Smets and Willem, and it is also called "codimension one symmetry" or "cap symmetry" by other authors. We refer the reader to the survey article [18] and the references therein for a broader discussion on symmetry properties of this type and its relationship to reflection inequalities. Finally we set

$$I := \overline{\{|x| \ : \ x \in B\}}.$$

The following is our main result for problem (1.1).

Theorem 1.1. *Suppose that*

2

(f1) the nonlinearity $f : [0, \infty) \times I \times \mathbb{R} \to \mathbb{R}$, $(t, r, u) \mapsto f(t, r, u)$ is continuous in t, r and locally Lipschitz in u uniformly with respect to t and r, i.e. for every $K > 0$ there is $L = L(K) > 0$ such that

$$|f(t, r, u_1) - f(t, r, u_2)| \leq L|u_1 - u_2|$$

for all $(t, r) \in [0, \infty) \times I$ and $u_1, u_2 \in [-K, K]$.

(f2) $f(\cdot, \cdot, 0)$ is bounded on $[0, \infty) \times I$.

Assume furthermore that $u \in C^{2,1}(B \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ is a classical solution of (1.1) such that

(U1) there is $e \in S$ such that $u_0 \geq u_0 \circ \sigma_e$ and $u_0 \not\equiv u_0 \circ \sigma_e$ in $B(e)$.

(U2) $\|u(\cdot, t)\|_{L^\infty(B)}$ is uniformly bounded in t.

Then u is asymptotically foliated Schwarz symmetric with respect to some $p \in S$, i.e. all elements in $\omega(u)$ are foliated Schwarz symmetric with respect to p.

Indeed, we will prove a more general version of this result in Section 2 below, dealing with a more general class of evolution problems similarly as in [13]. An immediate corollary of Theorem 1.1 is the following.

Corollary 1.2. *(i) Let $f : I \times \mathbb{R} \to \mathbb{R}$, $(r, u) \mapsto f(r, u)$ be continuous in $r \in I$ and locally Lipschitz in u uniformly with respect to r. Moreover, let $u \in C(\overline{B}) \cap C^2(B)$ be a classical solution of the elliptic problem*

$$\begin{aligned} -\Delta u &= f(|x|, u), && in \ B, \\ u(x) &= 0, && on \ \partial B, \end{aligned} \tag{1.3}$$

such that (U1) holds for u in place of u_0. Then u is foliated Schwarz symmetric with respect to some $p \in S$.

(ii) Suppose that $f : [0, \infty) \times I \times \mathbb{R} \to \mathbb{R}$ satisfies (f1) and is periodic in t, i.e. there is $T > 0$ such that $f(t + T, r, u) = f(t, r, u)$ for all t, r, u. Suppose furthermore that u is a T-periodic solution of (1.1), i.e., $u(x, t + T) = u(x, t)$ for all $x \in B, t \in [0, \infty)$, and such that (U1) holds. Then $u(\cdot, t)$ is foliated Schwarz symmetric with respect to some $p \in S$ for all times $t \in [0, \infty)$.

Both parts of this corollary are new. Under additional spectral assumptions on the solution, statements similar to part (i) have been derived in [8, 9] as an intermediate step in the proof of symmetry results for solutions of (1.3) with low Morse index. For time periodic solutions as considered in (ii), no previous symmetry result seems to be available in the present setting. We note that results on radial symmetry of nonnegative time periodic solutions had been obtained by Dancer and Hess [6] in the setting where B is a ball in $\mathbb{R}^N$ and f is nonincreasing in $|x|$.

We note that an easy example giving rise to a (sign changing) nonradial but foliated Schwarz symmetric solution of (1.3) – and thus also of (1.1) – is given by $f(|x|, u) = \lambda_2 u$, where λ_2 is the second Dirichlet eigenvalue of the Laplacian. It is known that every corresponding eigenfunction is of the form $u(x) = j(|x|)\frac{x \cdot p}{|x|}$ for some $p \in S$ and some positive function j on I, so u is obviously nonradial but foliated Schwarz symmetric with respect to p.

Our approach to prove (a more general version of) Theorem 1.1 is by a rotating plane argument, which should be seen as a variant of the moving plane method. The hypothesis (U1) will allow us to start this method. In contrast to the usual moving plane method on bounded domains in the form developed in [7] for elliptic and in [13] for parabolic problems, the symmetry axis is not fixed a priori by assumption (U1). Moreover, the rotating plane method alone only gives rise to local monotonicity with respect to every (cylindrical) angle. An extra argument is needed to translate this information into foliated Schwarz symmetry, see Proposition 3.3 below. Note also that assumption (U1) does not imply that the functions in $\omega(u)$ are strictly decreasing in the polar angle from the symmetry axis. For instance, in case B is a ball, f is decreasing in $|x|$ and $u_0 \in C(\overline{B})$ is a nonnegative function satisfying (U1), the above-mentioned result [13, Corollary 2.6] of Poláčik yields that $\omega(u)$ only consists of radial functions.

In the elliptic setting, the rotating plane method was used in combination with other arguments by Pacella and the second author [9] to prove – under some convexity hypothesis on the nonlinearity – foliated Schwarz symmetry of solutions with low Morse index. Later, this result was extended in [8] to unbounded domains under additional restrictions. The rotating plane method in the elliptic setting relies on different forms of the maximum principle (e.g., the maximum principle for small domains, see [2]). In the parabolic setting, the argument relies in a more subtle way on Harnack type inequalities and related estimates for linear equations. These estimates have been developed in a very useful form by Poláčik in order to derive asymptotic symmetry results in the Steiner symmetric setting [13], and we will make use of them in the present framework.

The paper is organized as follows. In Section 2 we present the general framework for our symmetry results in the context of fully nonlinear equations. In Section 3 we provide a new characterization of foliated Schwarz symmetry which is useful in combination with the rotating plane method. In Section 4, we recall some estimates for linear parabolic equations derived by Poláčik in [13], and we introduce a family of linear parabolic problems associated with the nonlinear problem. Finally, in Section 5 we apply the rotating plane method to the parabolic problem and prove the main result.

We add some closing remarks. Although Corollary 1.2 is an immediate consequence of Theorem 1.1, it can also be derived independently by a somewhat simpler argument not relying on the deep estimates in [13]. In order to keep this paper short, we leave the details to the reader.

In the present paper, we always consider a bounded radial domain. It is natural

to ask whether similar results are available in the case where $B = \mathbb{R}^N$ or B is the exterior of a ball in $\mathbb{R}^N$. This is part of work in progress. We note that, in a somewhat restricted setting, Poláčik [12] also developed a parabolic version of the moving plane method for the case where the underlying domain is the entire space. However, it is not straightforward to extend the parabolic *rotating plane argument* to the unbounded setting, since additional obstacles arise. In particular it seems more difficult than in [12] to start the method and to analyze extremal hyperplanes, since the local behaviour of f close to zero cannot be used in the same way as in [12].

2   The framework

In this section we set up a more general framework for our symmetry results. The setting is strongly inspired by [13]. We consider the fully nonlinear parabolic problem

$$
\begin{aligned}
u_t &= F(t, x, u, \nabla u, D^2 u), & (x,t) &\in (0, \infty) \times B, \\
u(x,t) &= 0, & (x,t) &\in \partial B \times (0, \infty), \\
u(x,0) &= u_0(x), & x &\in B,
\end{aligned}
\tag{2.1}
$$

where, as before, B is a bounded radial domain in $\mathbb{R}^N$, $N \geq 2$, and $D^2 u = (u_{x_i x_j})_{i,j=1}^N \in \mathbb{R}^{N \times N}$ is the Hessian of u. As for the right hand side of (2.1), we consider the following assumptions.

(F1) *Reflection invariance:* We have

$$
F : [0, \infty) \times \overline{B} \times \mathcal{B} \to \mathbb{R},
$$

where $\mathcal{B}$ is an open convex set in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N \times N}$ such that $B \times \mathcal{B}$ is invariant under the transformations

$(x, u, p, q) \mapsto (Rx, u, Rp, RqR)$, for every hyperplane reflection $R \in \mathbb{R}^{N \times N}$.

Moreover,
$$
F(t, Rx, u, Rp, RqR) = F(t, x, u, p, q)
$$
for every hyperplane reflection $R \in \mathbb{R}^{N \times N}$ and $(t, x, u, p, q) \in (0, \infty) \times B \times \mathcal{B}$.

(F2) *Regularity:* F is continuous on $[0, \infty) \times \overline{B} \times \mathcal{B}$ and Lipschitz in (u, p, q), uniformly with respect to x and t, i.e., there is $L > 0$ such that

$$
\sup_{x \in B, t \geq 0} |F(t, x, u, p, q) - F(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq L |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})|
$$

for all $(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{B}$. Moreover, F is differentiable with respect to q on $[0, \infty) \times B \times \mathcal{B}$.

(F3) *Boundedness:* $(0, 0, 0) \in \mathcal{B}$ and the function $F(\cdot, \cdot, 0, 0, 0)$ is bounded on $[0, \infty) \times B$.

(F4) *Ellipticity:* There is a constant $\alpha_0 > 0$ such that

$$\partial_{q_{ij}} F(t, x, u, p, q)\xi_i\xi_j \geq \alpha_0 |\xi|^2$$

for all $(t, x, u, p, q) \in [0, \infty) \times B \times \mathcal{B}$ and $\xi \in \mathbb{R}^N$. Here and below, we use the summation convention (summation over repeated indices).

We point out that these hypothesis are closely related to the ones in [13, Section 2]. However, in contrast to [13] we make no monotonicity assumptions on the nonlinearity and it may also include terms depending on the radial derivative of u. So this allows us to also consider equations like

$$u_t = g(t, |x|, u, |\nabla u|, \Delta u) + d(|x|)\nabla u \cdot x, \quad (x, t) \in B \times [0, \infty),$$

where $r \mapsto d(r)$ is a continuous function on $\mathbb{R}$, $g = g(t, r, u, \eta, \xi)$ is continuous on $\mathbb{R}^5$ and Lipschitz in (u, η, ξ) uniformly in (t, r), g_ξ exist everywhere and $g_\xi \geq \alpha_0$ for some positive constant α_0.

The symmetry result which we want to prove in this general setting relies also on assumptions $(U1)$ and $(U2)$ for a fixed solution of (2.1), which were stated in the introduction.

Theorem 2.1. *Assume $(F1)-(F4)$, and let $u \in C^{2,1}(B \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be a classical solution of the problem (2.1) satisfying assumptions $(U1)$ and $(U2)$. Then u is asymptotically foliated Schwarz symmetric with respect to some $p \in S$, i.e. all the elements in $\omega(u)$ are foliated Schwarz symmetric with respect to p.*

We quickly show how Theorem 2.1 implies Theorem 1.1. Indeed, by (U2), we have

$$K := \sup_{t \geq 0, x \in B} |u(x, t)| < \infty.$$

Hence we may consider (1.1) as a special case of (2.1) with $\mathcal{B} = (-K - 1, K + 1) \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ and

$$F : [0, \infty) \times \overline{B} \times \mathcal{B} \to \mathbb{R}, \qquad F(t, x, u, p, q) = \text{trace}(q) + f(t, |x|, u).$$

With this definition, assumptions (F1) and (F4) are obviously satisfied. Moreover, (F2) and (F3) follow from assumptions (f1) and (f2) of Theorem 1.1, respectively. Hence the assumptions of Theorem 1.1 imply those of Theorem 2.1, and therefore Theorem 1.1 follows.

We point out that, as noted in [13, Proposition 2.7], assumptions (F2),(F3) and (F4) on the nonlinearity F ensure that, for every solution $u \in C^{2,1}(B \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ of (2.1) satisfying (U2),

$$\sup_{\substack{x, \bar{x} \in B, t, \bar{t} \in [s, s+1], \\ x \neq \bar{x}, t \neq \bar{t}, \\ s \geq 1}} \frac{|u(x, t) - u(\bar{x}, \bar{t})|}{|x - \bar{x}|^\alpha + |t - \bar{t}|^{\frac{\alpha}{2}}} < \infty \qquad \text{for some } \alpha > 0. \qquad (2.2)$$

Indeed, being a bounded radial domain, B is smoothly bounded and therefore satisfies assumption (A) of [13, Proposition 2.7]. It follows immediately from

6

(2.2) that the orbit $\{u(\cdot, t) \ : \ t > 0\}$ is relatively compact in $C_0(\overline{B})$ and that $\omega(u)$ is a nonempty compact subset of $C_0(\overline{B})$ satisfying $\mathrm{dist}(u(t, \cdot), \omega(u)) \to 0$ in $C_0(\overline{B})$ as $t \to \infty$.

We finally note that hypothesis (F3) can be dropped if we assume instead a priori that $\{u(\cdot, t) : t > 0\}$ is an equicontinuous subset of $C(\overline{B})$.

3 Characterizations of foliated Schwarz symmetry

As before, B denotes a radial subdomain of $\mathbb{R}^N$, $N \geq 2$, and I, S, $H(e)$, $B(e)$ and $\sigma(e)$ are defined as in the introduction for $e \in S$. We start by proving an auxiliary lemma.

Lemma 3.1. *Let $v \in C(\mathbb{R})$ be an even and 2π-periodic function, and let $\mathcal{R}$ denote the points of reflectional symmetry of v. If, for some $\eta \in \mathbb{R}$,*

$$\begin{aligned} v(\eta + \varphi) &\geq v(\eta - \varphi) \quad && \text{for all } \varphi \in [0, \pi] \text{ and} \\ v(\eta + \varphi_0) &> v(\eta - \varphi_0) \quad && \text{for some } \varphi_0 \in (0, \pi). \end{aligned} \tag{3.1}$$

then we have $\mathcal{R} = \{n\pi \ : \ n \in \mathbb{Z}\}$.

Proof. ¿From the fact that v is even, continuous, 2π-periodic and non-constant by assumption, it is easy to deduce that $\mathcal{R} = \{\frac{n\pi}{k} \ : \ n \in \mathbb{Z}\}$ with some positive integer k. We suppose by contradiction that $k \geq 2$. Then v is $\frac{2\pi}{k}$−periodic, and for a suitable translation w of v we can assume that there is some $\eta \in (0, \frac{\pi}{k})$ and some $\varphi_0 \in (0, \frac{2\pi}{k}]$ such that

$$\begin{aligned} w(\varphi) &= w(-\varphi) && \text{for all } \varphi \in \mathbb{R}, \\ w\left(\pm \frac{\pi}{k} + \varphi \right) &= w\left(\pm \frac{\pi}{k} - \varphi \right) && \text{for all } \varphi \in \mathbb{R}, \\ w(\eta + \varphi) &\geq w(\eta - \varphi) && \text{for all } \varphi \in (0, \pi), \\ w(\eta + \varphi_0) &> w(\eta - \varphi_0). \end{aligned}$$

Since $0 < \frac{2\pi}{k} - \varphi_0 < \pi$, it follows that

$$w(\eta + \varphi_0) > w(\eta - \varphi_0) = w\left(-\frac{2\pi}{k} - (\eta - \varphi_0) \right) = w\left(\eta + \frac{2\pi}{k} - \varphi_0 \right)$$

$$\geq w\left(\eta - \frac{2\pi}{k} + \varphi_0 \right) = w\left(\frac{2\pi}{k} - \eta - \varphi_0 \right) = w(\eta + \varphi_0),$$

which yields a contradiction. Hence $k = 1$, and thus the claim follows. $\quad\square$

Now we generalize a result of Brock ([3], Lemma 4.2) to characterize sets of foliated Schwarz symmetric functions with respect to a common direction.

Proposition 3.2. *Let $\mathcal{U}$ be a set of continuous functions defined on a radial domain $B \subset \mathbb{R}^N$, $N \geq 2$. Define*

$$M := \{e \in S \mid u(x) \geq u(\sigma_e(x)) \text{ for all } x \in B(e) \text{ and } u \in \mathcal{U}\}. \qquad (3.2)$$

If

$$S = M \cup -M, \qquad (3.3)$$

i.e., if for all $e \in S$ we have

$$u \geq u \circ \sigma_e \text{ in } B(e) \text{ for all } u \in \mathcal{U} \qquad \text{or} \qquad u \leq u \circ \sigma_e \text{ in } B(e) \text{ for all } u \in \mathcal{U},$$

then there is $p \in S$ such that every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to p.

Proof. We start by constructing orthogonal unit vectors $e_1, \ldots, e_{N-1}$ such that

$$u \equiv u \circ \sigma_{e_i} \qquad \text{for } i = 1, \ldots, N-1 \text{ and every } u \in \mathcal{U}. \qquad (3.4)$$

For this we first consider the set

$$\mathcal{A}_1 := \{e \in S \ : \ u(x) > u(\sigma_e(x)) \text{ for some } u \in \mathcal{U} \text{ and some } x \in B(e)\}.$$

By (3.3) we have $\mathcal{A}_1 \subset M$, and $\mathcal{A}_1$ does not contain antipodal points. Moreover, $\mathcal{A}_1$ is a relatively open subset of S. If $\mathcal{A}_1$ is empty, then $u \equiv u \circ \sigma_e$ for any $u \in \mathcal{U}$ and $e \in S$, so any choice of orthonormal vectors $e_1, \ldots, e_{N-1}$ satisfies (3.4). Hence we may assume that $\mathcal{A}_1 \neq \varnothing$. Then also the relative boundary $\partial \mathcal{A}_1$ of $\mathcal{A}_1$ in S is non-empty. Let $e_1 \in \partial \mathcal{A}_1$; then any $u \in \mathcal{U}$ satisfies $u \equiv u \circ \sigma_{e_1}$. Next we consider

$$\mathcal{A}_2 := \{e \in S \cap H(e_1) \ : \ u(x) > u(\sigma_e(x)) \text{ for some } u \in \mathcal{U} \text{ and some } x \in B(e)\}.$$

If $\mathcal{A}_2$ is empty, then may complement e_1 with any choice of orthonormal vectors $e_2, \ldots, e_{N-1}$ in $S \cap H(e_1)$ to obtain (3.4). If $\mathcal{A}_2$ is nonempty, then – by the same argument as above – also the relative boundary $\partial \mathcal{A}_2$ of $\mathcal{A}_2$ in $S \cap H(e_1)$ is nonempty, and every vector $e_2 \in \partial \mathcal{A}_2$ satisfies $u \equiv u \circ \sigma_{e_2}$ for every $u \in \mathcal{U}$. Successively we find orthogonal vectors $e_1, \ldots, e_{N-1} \in S$ such that (3.4) holds (then the process stops since $S \cap H(e_1) \cap H(e_2) \cap \cdots \cap H(e_{N-1})$ consists merely of two antipodal points).

Without loss of generality, we may now assume that the vectors $e_1, \ldots, e_{N-1}$ satisfying (3.4) are the first $N-1$ coordinate vectors. Next we show that every hyperplane containing the x_N-axis is a symmetry hyperplane for every $u \in \mathcal{U}$. For this let $q = (q_1, \ldots, q_N) \in S$ be such that $\mathbb{R}e_N \subset H(q)$. By (3.3) we can assume that $q \in M$ (otherwise we replace q by $-q$). Since $q_N = 0$, for $x \in B(q)$ we have that $[\sigma_{e_1} \circ \ldots \circ \sigma_{e_{N-1}}](x) = -\sigma_{e_N}(x) \notin B(q)$, and from (3.4) we deduce that

$$u(x) = u(-\sigma_{e_N}(x)) \leq u(\sigma_q(-\sigma_{e_N}(x))) = u(-\sigma_{e_N}(\sigma_q(x))) = u(\sigma_q(x)) \leq u(x)$$

for every $u \in \mathcal{U}$. Hence $u \equiv u \circ \sigma_q$ for every $u \in \mathcal{U}$, as claimed. We conclude that every $u \in \mathcal{U}$ is axially symmetric with respect to the axis $\mathbb{R}e_N$.

To complete the proof of foliated Schwarz symmetry, we may now restrict to any two-dimensional subspace of $\mathbb{R}^N$ containing the axis $\mathbb{R}e_N$, hence we may assume that $N = 2$ from now on. Let $u \in \mathcal{U}$ be a non radial function. Then there are $e_* \in S$ and $x \in B(e_*)$ such that $e_* \cdot e_2 > 0$ and

$$u(x) > u(\sigma_{e_*}(x)) \text{ or} \qquad\qquad (3.5)$$
$$u(x) < u(\sigma_{e_*}(x)). \qquad\qquad (3.6)$$

Assume (3.5) first. Writing $u = u(r, \varphi)$ in (permuted) polar coordinates with $x_1 = r \sin \varphi$ and $x_2 = r \cos \varphi$, we get that u is even in φ, and that there are $r > 0$ and $\eta \in (-\pi, 0)$ such that (3.1) holds for the function $\mathbb{R} \to \mathbb{R}$, $\varphi \mapsto u(r, \varphi)$. Hence by Lemma 3.1 there are no other points of reflectional symmetry of this function in $(-\pi, 0)$ except the origin, and by (3.3) this implies that for every $e \in S$ with $e \cdot e_2 > 0$ we have $u \geq u \circ \sigma_e$ and $u \not\equiv u \circ \sigma_e$ in $B(e)$. Then again by (3.3) we have that

$$u \geq u \circ \sigma_e \text{ in } B(e) \text{ for all } u \in \mathcal{U} \text{ and all } e \in S \text{ with } e \cdot e_2 \geq 0,$$

and this readily implies that every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to the unit vector e_2.
A similar argument shows that, if we assume (3.6) then every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to the unit vector $-e_2$. The proof is finished. $\qquad\square$

The following Proposition characterizes foliated Schwarz symmetry by properties related to the method of rotating planes.

Proposition 3.3. *Let $\mathcal{U}$ be a set of continuous functions defined on a radial domain $B \subset \mathbb{R}^N$, $N \geq 2$, and let M be defined as in (3.2). Moreover, let $\tilde{e} \in M$. If for all two dimensional subspaces $P \subseteq \mathbb{R}^N$ containing $\tilde{e}$ there are two different points p_1, p_2 in the same connected component of $M \cap P$ such that $u \equiv u \circ \sigma_{p_1}$ and $u \equiv u \circ \sigma_{p_1}$ for every $u \in \mathcal{U}$, then there is $p \in S$ such that every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to p.*

Proof. Let P be a two dimensional subspace with $\tilde{e} \in P$. By hypothesis there is some connected component K_P of $M \cap P$ and $p_1, p_2 \in K_P$ such that $u \equiv u \circ \sigma_{p_1}$ and $u \equiv u \circ \sigma_{p_2}$ for every $u \in \mathcal{U}$. We first show that

$$K_P \text{ contains a closed halfcircle}, \qquad\qquad (3.7)$$

i.e., $\{e \in S \cap P : e \cdot e' \geq 0\} \subseteq K_P$ for some $e' \in S$. We assume without loss of generality that

$$p_1 = (1, 0, \ldots, 0), \qquad p_2 = (\cos \psi, \sin \psi, 0, \ldots, 0) \quad \text{for some } \psi \in (0, 2\pi]$$

and

$$(\cos \varphi, \sin \varphi, 0, \ldots, 0) =: p_\varphi \in M \qquad \text{for all } \varphi \in [0, \psi]$$

9

(because p_1 and p_2 are in the same connected component of $M \cap P$). Let $u \in \mathcal{U}$. Using polar coordinates, we define

$$\tilde{v}(r, \varphi, x') := u(r \cos \varphi, r \sin \varphi, x') = u(x)$$

with $x \in B$, $x' = (x_3, \ldots, x_N) \in \mathbb{R}^{N-2}$, $\varphi \in \mathbb{R}$, and $r = |x| \in I$. If, independently of the choice of $u \in \mathcal{U}$, $\tilde{v}$ does not depend on φ, then $M \cap P = S \cap P$ and so (3.7) holds trivially. So, we may suppose that $u \in \mathcal{U}$ was chosen such that the function

$$v : \mathbb{R} \to \mathbb{R}, \qquad v(\varphi) := \tilde{v}(r, \varphi, x')$$

is non-constant for some fixed $r > 0$ and $x' \in \mathbb{R}^{N-2}$. By assumption, we then have

$$
\begin{aligned}
v(\varphi) &= v(-\varphi), & \varphi &\in \mathbb{R}, \\
v(\psi + \varphi) &= v(\psi - \varphi), & \varphi &\in \mathbb{R}, \\
v(\eta + \varphi) &\geq v(\eta - \varphi), & \eta &\in (0, \psi), \varphi \in (0, \pi), \qquad (3.8)
\end{aligned}
$$

i.e. v has two points of reflectional symmetry, one at zero, and one at ψ, and the points in between satisfy the defining property of M. Since the function is non-constant, the inequality in (3.8) must be strict for some $\eta \in (0, \psi)$ and $\varphi \in (0, \pi)$. Then, by Lemma 3.1, we get that $u \not\equiv u \circ \sigma_{p_\varphi}$ for $\varphi \in (0, \pi)$. By assumption, we then conclude that $p_2 \neq p_\varphi$ for $\varphi \in (0, \pi)$, and therefore $\psi \geq \pi$. Hence (3.7) holds, as claimed.

Now since (3.7) holds independently of P, we conclude that, for all $e \in S$ we have $e \in M$ or $-e \in M$, so that (3.3) holds. Hence the assertion follows from Proposition 3.2. $\qquad \square$

4 Linear parabolic problems associated with reflections at hyperplanes

To use the rotating plane method in the parabolic setting, the crucial step is to consider the linear problem satisfied by the difference between a solution of (2.1) and its reflection at a hyperplane. In order to deal with this problem, we first quote estimates derived by Poláčik [13] for linear parabolic equations in a general setting. So in the following, we consider the general linear equation

$$v_t = a_{ij}(x,t)v_{x_i x_j} + b_i(x,t)v_{x_i} + c(x,t)v, \quad (x,t) \in U \times (\tau, T), \qquad (4.1)$$
$$v = 0, \quad (x,t) \in \partial U \times (\tau, T), \qquad (4.2)$$

where U is an open subset of some fixed bounded domain $\Omega \subset \mathbb{R}^N$, $0 \leq \tau < T \leq \infty$, the coefficients a_{ij}, b_i, c are defined on $U \times (\tau, T)$, are measurable and for some positive constants α_0, β_0 satisfy that

$$
\begin{aligned}
|a_{ij}(x,t)|, |b_i(x,t)|, |c(x,t)| &< \beta_0, & x \in U, t \in [\tau, T), i, j = 1, \ldots, N, \\
a_{ij}(x,t)\xi_i \xi_j &\geq \alpha_0 |\xi|^2, & x \in U, t \in [\tau, T), \xi \in \mathbb{R}^N.
\end{aligned}
\qquad (4.3)
$$

When referring to a solution of equation (4.1), we mean a function v in the Sobolev space $W^{2,1}_{N+1,loc}(U \times (\tau, T))$ such that (4.1) is satisfied almost everywhere. A solution of the boundary value problem (4.1),(4.2) is in addition supposed to be continuous on $\overline{U} \times [\tau, T]$ and to satisfy (4.2) in the pointwise sense. The following two results are special cases of Theorems by Poláčik (see [13, Lemma 3.4 and Theorem 3.7]).

Lemma 4.1. *(Special case of [13, Lemma 3.4])*
Let Ω be a bounded domain. Given $d, \theta > 0$, there is a positive constant κ determined only by $N, \operatorname{diam}(\Omega), \alpha_0, \beta_0, d$ and θ with the following property. If D, U are domains in Ω with $D \subset\subset U$, $\operatorname{dist}(\overline{D}, \partial U) \geq d$, and $v \in C(\overline{U} \times [\tau, \tau + 4\theta])$ is a solution of (4.1), (4.2) on $U \times (\tau, \tau + 4\theta)$, then

$$\inf_{D \times (\tau + 3\theta, \tau + 4\theta)} v \geq \kappa \sup_{D \times (\tau + \theta, \tau + 2\theta)} v - e^{4m\theta} \sup_{\partial_P(U \times (\tau, \tau + 4\theta))} v^-,$$

where $m = \sup\limits_{U \times (\tau, \tau + 4\theta)} c.$

Here $v^+ := \max\{v, 0\}$ and $v^- := -\min\{v, 0\}$ denote the usual positive and negative parts of a function v. Moreover, $\partial_P(U \times (\tau, \tau + 4\theta)) = \overline{U} \times \{\tau\} \cup \partial U \times (\tau, T)$ denotes the parabolic boundary of $U \times (\tau, \tau + 4\theta)$. In the following, we also use the notation

$$\operatorname{inrad}(\Omega) := \sup\{r > 0 : B_r(x) \subset \Omega \text{ for some } x \in \Omega\},$$

where $B(x, r) = B_r(x) = \{y \in \mathbb{R}^N : |x - y| < r\}$.

Theorem 4.2. *(Special case of [13, Theorem 3.7])*
Fix $\rho \in (0, \frac{\operatorname{diam}(\Omega)}{2})$. Then there is

$$\delta = \delta(N, \operatorname{diam}(\Omega), \alpha_0, \beta_0, \rho) > 0$$

and, for every $d, \theta > 0$,

$$\mu = \mu(N, \operatorname{diam}(\Omega), \alpha_0, \beta_0, d, \theta, \rho) \in (0, 1]$$

with the following properties: If $D \subset U$ are subdomains of Ω satisfying

$$\operatorname{inrad}(D) > \rho, \quad |U \backslash \overline{D}| < \delta,$$
$$\operatorname{dist}(\overline{D}, \partial U) > d,$$

if $v \in C(\overline{U} \times [\tau, \infty))$ is a solution of a problem (4.1),(4.2) whose coefficients satisfy (4.3) (with $T = \infty$), and if

$$v(x, t) > 0 \quad \text{for } (x, t) \in \overline{D} \times [\tau, \tau + 8\theta),$$
$$\|v^-(\cdot, \tau)\|_{L^\infty(U \backslash D)} \leq \mu \|v\|_{L^\infty(D \times (\tau + \theta, \tau + 2\theta))}, \tag{4.4}$$

then the following statements hold true:

(S1) $v(x,t) > 0$ for all $(x,t) \in \overline{D} \times [\tau, \infty)$.

(S2) $\|v^-(x,t)\|_{L^\infty(U)} \to 0$, as $t \to \infty$.

We now come back to the linear problem satisfied by the difference between a solution of (2.1) and its reflection at a hyperplane. So as before, B denotes a radial subdomain of $\mathbb{R}^N$, $N \geq 2$, and I, S, $H(e)$, $B(e)$ and $\sigma(e)$ are defined as in the introduction for $e \in S$. Moreover, we let u denote a solution of (2.1), and for $e \in S$ we define

$$w_e(x,t) := u(x,t) - u(\sigma_e(x),t) \qquad \text{for } (x,t) \in B(e) \times [0,\infty).$$

Then w_e is a solution of the problem

$$
\begin{aligned}
\partial_t w_e &= a_{ij}^e(x,t)(w_e)_{x_i x_j} + b_i^e(x,t)(w_e)_{x_i} + c^e(x,t)w_e, & (x,t) \in B(e) \times (0,\infty), \\
w_e(x,t) &= 0, & (x,t) \in \partial B(e) \times (0,\infty), \\
w_e(x,0) &= u_0(x) - u_0(\sigma_e(x)), & x \in B(e),
\end{aligned}
$$
(4.5)

where the coefficients are obtained, as in [13], via the Hadamard formula. To make this precise, let $u^e(x,t) := u(\sigma_e(x),t)$ and consider

$$
c^e(x,t) := \begin{cases} \int_0^1 F_u(t,|x|, su + (1-s)u^e, Du, D^2 u))ds, & \text{if } u(x,t) \neq u^e(x,t), \\ 0, & \text{if } u(x,t) = u^e(x,t), \end{cases}
$$

$$
b_i^e(x,t) := \begin{cases} \int_0^1 F_{p_i}(t,|x|, u^e, \dots, u^e_{x_{i-1}}, su_{x_i} \\ \qquad + (1-s)u^e_{x_i}, u_{x_{i+1}}, \dots, D^2 u))ds, & \text{if } u_{x_i}(x,t) \neq u^e_{x_i}(x,t), \\ 0, & \text{if } u_{x_i}(x,t) = u^e_{x_i}(x,t), \end{cases}
$$

$$
a_{ij}^e(x,t) := \int_0^1 F_{q_{ij}}(t,|x|, u^e, Du^e, \dots, u^e_{x_{i^-} x_{j^-}}, su_{x_i x_j} \\ \qquad + (1-s)u^e_{x_i x_j}, u_{x_{i^+} x_{j^+}}, \dots, u_{x_N x_N}))ds,
$$

where $(i^-, j^-), (i^+, j^+)$ stand for the pairs of indices preceding, respectively, following, (i,j) within a fixed identification of $\mathbb{R}^{N \times N}$ with $\mathbb{R}^{N^2}$.

By (F1) and (F2) the integrals make sense and give the right quotients for the right hand side of (4.5) to be equal to the difference of $F(t,|x|, u, Du, D^2 u)$ and $F(t,|x|, u^e, Du^e, D^2 u^e)$.

For every $z \in \omega(u)$, let

$$z_e \in C_0(\overline{B(e)}), \qquad z_e(x) := z(x) - z(\sigma_e(x)) \text{ for } x \in B(e).$$

Finally we define the set

$$\mathcal{M} := \{e \in S \mid z_e(x) \geq 0 \text{ for all } x \in B(e) \text{ and } z \in \omega(u)\}. \qquad (4.6)$$

We remark that, as a consequence of (F2) and (F4), there is $\beta_0 > 0$ such that

$$|c^e(x,t)|, |b_i^e(x,t)|, |a_{ij}^e(x,t)| < \beta_0 \qquad \text{and} \qquad a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \qquad (4.7)$$

for all $(x,t) \in B(e) \times [0,\infty), i,j = 1, \dots, N, \xi \in \mathbb{R}^N$ and $e \in S$ with $\alpha_0 > 0$ as in (F4).

5 Proof of the main result

As before, B denotes a radial subdomain of $\mathbb{R}^N$, $N \geq 2$, and I, S, $H(e)$, $B(e)$ and $\sigma(e)$ are defined as in the introduction for $e \in S$. Moreover, for a fixed solution u of (2.1) satisfying the assumptions of Theorem 2.1, we will make use of the definitions introduced in Section 4. Recall, in particular, the definition of $\mathcal{M}$ in (4.6).

Lemma 5.1. *Let $e \in S$ be as in (U1). Then there is some $\varepsilon > 0$ such that $e' \in \mathcal{M}$ for all $e' \in S$ with $|e' - e| < \varepsilon$.*

Proof. If $e \in S$ is as in (U1), then it follows from (4.5) and the parabolic strong maximum principle (see for example [14]) that

$$w_e(x,t) > 0 \text{ in } B(e) \times (0,\infty), \tag{5.1}$$

and therefore $e \in \mathcal{M}$. Let $\delta > 0$ be chosen as in Theorem 4.2 corresponding to $\Omega = B$, $\rho := \frac{\text{inrad}(B)}{4}$, and α_0, β_0 as in (4.7). Moreover, let $D \subset\subset B(e)$ be a subdomain such that $|B(e)\backslash D| < \delta$ and $\text{inrad}(D) > \rho$. Put $d := \frac{\text{dist}(\overline{D}, \partial B(e))}{2}$, $\theta := 1$, and let $\mu \in (0,1]$ be as in Theorem 4.2 (corresponding to these choices of $\Omega, \alpha_0, \beta_0, d, \theta$ and ρ). By (5.1) there exists some $\eta > 0$ such that

$$w_e(x,t) > \eta > 0, \quad (x,t) \in \overline{D} \times [1,9].$$

Moreover, there is some $\varepsilon > 0$ such that for all $e' \in S$ with $|e - e'| < \varepsilon$ we have

$$D \subset\subset B(e'), \qquad |B(e')\backslash D| < \delta, \qquad \text{dist}(\overline{D}, \partial B(e')) > d$$

and, as a consequence of continuity and (5.1),

$$w_{e'}(x,t) > \frac{\eta}{2} > 0, \quad (x,t) \in \overline{D} \times [1,9],$$

$$\|w_{e'}^-(\cdot,1)\|_{L^\infty(B(e'))} \leq \frac{\mu\eta}{2} \leq \mu\|w_{e'}\|_{L^\infty(D \times [2,3])}.$$

Hence for these $e' \in S$ the hypothesis of Theorem 4.2 are satisfied with $U = B(e')$, $\tau = 1$, $\theta = 1$ and D as above, and thus we get that

$$\|w_{e'}^-(\cdot,t)\|_{L^\infty(B(e'))} \to 0, \text{ as } t \to \infty.$$

This shows $e' \in \mathcal{M}$ for $e' \in S$ with $|e - e'| < \varepsilon$, as claimed. $\qquad\square$

Lemma 5.2. *Let $e \in \mathcal{M}$. If there is some $\tilde{z} \in \omega(u)$ such that $\tilde{z}_e \not\equiv 0$, then there is some $\varepsilon > 0$ such that $e' \in \mathcal{M}$ for all $e' \in S$ with $|e - e'| < \varepsilon$.*

Proof. Since $\tilde{z}_e \not\equiv 0$ there is some $\alpha > 0$ and $x_0 \in B(e)$ such that $\tilde{z}_e(x_0) \geq 2\alpha > 0$. Let $\delta > 0$ be chosen as in Theorem 4.2 corresponding to $\Omega = B$, $\rho := \frac{\text{inrad}(B)}{4}$, and α_0, β_0 as in (4.7). Moreover, let $D \subset\subset B(e)$ be a subdomain such that $|B(e)\backslash D| < \delta$, $\text{inrad}(D) > \rho$ and $x_0 \in D$. Put $d := \frac{\text{dist}(\overline{D}, \partial B(e))}{2}$, $\theta := \frac{1}{8}$, and let

$\mu \in (0, 1]$ be as in Theorem 4.2 (corresponding to these choices of $\Omega, \alpha_0, \beta_0, d, \theta$ and ρ). Since $z_e \geq 0$ in $B(e)$ for all $z \in \omega(u)$ and, as remarked at the end of Section 2, $\text{dist}(u(\cdot, t), \omega(u)) \to 0$ in $C_0(\overline{B})$ as $t \to \infty$, there is some $T_0 > 0$ such that

$$\|w_e^-(\cdot, t)\|_{L^\infty(B(e))} < \frac{\mu \kappa \alpha}{8} e^{-4\beta_0} \qquad \text{for } t \geq T_0, \tag{5.2}$$

where $\kappa > 0$ is the constant given by Lemma 4.1 for $\Omega, \alpha_0, \beta_0, d$ as above and $\theta = 1$. Next, we may pick $T_1 \geq T_0 + 1$ such that $\|w_e(\cdot, T_1) - \tilde{z}_e\|_{L^\infty(B(e))} < \alpha$ and therefore $w_e(x_0, T_1) > \alpha$. We then apply Lemma 4.1 to $U = B(e)$, $\tau := T_1 + 2$ and $\theta = 1$ in order to get

$$\inf_{D \times (\tau, \tau+1)} w_e \geq \kappa \|w_e^+\|_{L^\infty(D \times (\tau-2, \tau-1))} - e^{4\beta_0} \sup_{\partial_P(B(e) \times (\tau-3, \tau+1))} w_e^-$$

$$\geq \kappa \alpha - \frac{\mu \kappa \alpha}{8} \geq \frac{\kappa \alpha}{2} =: \eta > 0.$$

Moreover, there is some $\varepsilon > 0$ such that for all $e' \in S$ with $|e - e'| < \varepsilon$ we have

$$D \subset\subset B(e'), \qquad |B(e') \backslash D| < \delta, \qquad \text{dist}(\overline{D}, \partial B(e')) > d$$

and, by continuity,

$$\inf_{D \times (\tau, \tau+1)} w_{e'} \geq \frac{\eta}{2} \qquad \text{and} \qquad \|w_{e'}^-(\cdot, \tau)\|_{L^\infty(B(e'))} \leq \|w_e^-(\cdot, \tau)\|_{L^\infty(B(e))} + \frac{\eta \mu}{4}.$$

Combining this with (5.2), we find that

$$\|w_{e'}^-(\cdot, \tau)\|_{L^\infty(B(e'))} \leq \frac{\eta \mu}{4} + \frac{\mu \kappa \alpha}{8} e^{-4\beta_0} \leq \frac{\mu \eta}{2} \leq \mu \|w_{e'}^+\|_{L^\infty(D \times (\tau+\frac{1}{8}, \tau+\frac{1}{4}))}$$

for every $e' \in S$ with $|e - e'| < \varepsilon$. In particular, for these $e' \in S$ the hypothesis of Theorem 4.2 are satisfied with $U = B(e')$ and $\theta = \frac{1}{8}$, and therefore

$$\|w_{e'}^-(x, t)\|_{L^\infty(B(e'))} \to 0 \quad \text{as } t \to \infty.$$

This yields $e' \in \mathcal{M}$ for all $e' \in S$ with $|e - e'| < \varepsilon$, as claimed. $\qquad \square$

We are now ready to prove the main symmetry result.

Proof of Theorem 2.1. Let $e \in S$ be as in (U1). Then, by Lemma 5.1, there is $\varepsilon > 0$ such that

$$e' \in \mathcal{M} \qquad \text{for all } e' \in S \text{ with } |e' - e| < \varepsilon. \tag{5.3}$$

Let P be any two dimensional subspace of $\mathbb{R}^N$ containing e. Without loss of generality, we may assume that $e = (1, 0, \ldots, 0)$ and $P = \{x = (x_1, 0, ..., 0, x_N) \mid x_1, x_N \in \mathbb{R}\}$. Define

$$e_\theta := (\cos(\theta), 0, ..., 0, \sin(\theta)), \qquad z_\theta := z_{e_\theta} \in \mathcal{C}_0(B(e_\theta))$$

14

for $\theta \in \mathbb{R}$ and

$$\Theta_1 := \sup\{\theta > 0 \: : \: e_\varphi \in \mathcal{M} \text{ for all } 0 \leq \varphi \leq \theta\},$$
$$\Theta_2 := \inf\{\theta < 0 \: : \: e_\varphi \in \mathcal{M} \text{ for all } \theta \leq \varphi \leq 0\}.$$

We note that $\Theta_2 < 0 < \Theta_1$ by (5.3). If $\Theta_1 - \Theta_2 \geq 2\pi$ (and in particular if $\Theta_1 = \infty$ or $\Theta_2 = -\infty$), it immediately follows from the definition of $\mathcal{M}$ that every $H(e_\theta)$, $\theta \in \mathbb{R}$, is a symmetry hyperplane for all the elements in $\omega(u)$. If both Θ_1 and Θ_2 are finite and $\Theta_1 - \Theta_2 < 2\pi$, we have $z_{\Theta_1} \equiv z_{\Theta_2} \equiv 0$ for all $z \in \omega(u)$ as a consequence of Lemma 5.2, so that $H(e_{\Theta_1})$ and $H(e_{\Theta_2})$ are symmetry hyperplanes for all the elements in $\omega(u)$. Moreover, $e_{\Theta_1} \neq e_{\Theta_2}$ and $e_\varphi \in \mathcal{M}$ for all $\varphi \in (\Theta_2, \Theta_1)$. Since this can be done for all two dimensional subspaces P of $\mathbb{R}^N$ containing e, we can use Proposition 3.3, applied to $\mathcal{U} = \omega(u)$, to obtain the existence of $p \in S$ such that every $z \in \omega(u)$ is foliated Schwarz symmetric with respect to p, as claimed. □

6 Acknowledgements

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